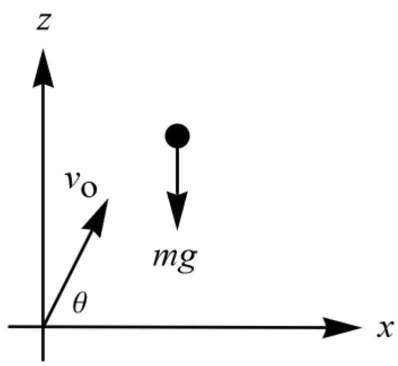
Problem 1.35

A golf ball is hit from ground level with speed v_o in a direction that is due east and at an angle θ above the horizontal. Neglecting air resistance, use Newton's second law (1.35) to find the position as a function of time, using coordinates with x measured east, y north, and z vertically up. Find the time for the golf ball to return to the ground and how far it travels in that time.

Solution

Start by drawing a free-body diagram of the golf ball. Because there's no air resistance, there's only a gravitational force acting on the ball.



Newton's second law states that the sum of the forces on the ball is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{cases}$$

The only force is due to gravity, and it's in the negative z-direction.

$$\begin{cases} 0 = ma_x \\ 0 = ma_y \\ -mg = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} 0 = a_x \\ 0 = a_y \\ -g = a_z \end{cases}$$

Acceleration is the second derivative of position.

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = 0 \\ \dfrac{d^2z}{dt^2} = -g \end{cases}$$

Integrate both sides of each equation with respect to t to get the components of the ball's velocity.

$$\begin{cases} \dfrac{dx}{dt} = C_1 \\ \dfrac{dy}{dt} = C_2 \\ \dfrac{dz}{dt} = -gt + C_3 \end{cases} \tag{1}$$

The initial velocity in the x-, y-, and z-directions are $v_\text{o} \cos\theta$, 0, and $v_\text{o} \sin\theta$, respectively.

$$\begin{aligned} \frac{dx}{dt}(0) &= C_1 = v_\text{o}\cos\theta & \rightarrow \quad C_1 &= v_\text{o}\cos\theta \\ \frac{dy}{dt}(0) &= C_2 = 0 & \rightarrow \quad C_2 &= 0 \\ \frac{dz}{dt}(0) &= -g(0) + C_3 = v_\text{o}\sin\theta & \rightarrow \quad C_3 &= v_\text{o}\sin\theta \end{aligned}$$

As a result, equation (1) becomes

$$\begin{cases} \dfrac{dx}{dt} = v_\text{o}\cos\theta \\ \dfrac{dy}{dt} = 0 \\ \dfrac{dz}{dt} = -gt + v_\text{o}\sin\theta \end{cases}.$$

Integrate both sides of each equation with respect to t once more to get the components of the ball's position.

$$\begin{cases} x(t) = v_\text{o}t\cos\theta + C_4 \\ y(t) = C_5 \\ z(t) = -\dfrac{gt^2}{2} + v_\text{o}t\sin\theta + C_6 \end{cases} \tag{2}$$

The ball's initial position is the origin, so $x = 0$, $y = 0$, and $z = 0$ when $t = 0$.

$$\begin{aligned} x(0) &= v_\text{o}(0)\cos\theta + C_4 = 0 & \rightarrow \quad C_4 &= 0 \\ y(0) &= C_5 = 0 & \rightarrow \quad C_5 &= 0 \\ z(0) &= -\frac{g(0)^2}{2} + v_\text{o}(0)\sin\theta + C_6 = 0 & \rightarrow \quad C_6 &= 0 \end{aligned}$$

Consequently, equation (2) becomes

$$\begin{cases} x(t) = v_o t\cos\theta \\ y(t) = 0 \\ z(t) = -\dfrac{gt^2}{2} + v_o t\sin\theta \end{cases}.$$

Therefore, the ball's position is

$$\boxed{\mathbf{r}(t) = \left\langle v_o t\cos\theta, 0, -\frac{gt^2}{2} + v_o t\sin\theta \right\rangle.}$$

To find how long the ball is in the air for, set $z(t) = 0$ and solve for nonzero t.

$$z(t) = -\frac{gt^2}{2} + v_o t\sin\theta = 0$$

$$t\left(-\frac{gt}{2} + v_o\sin\theta\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{gt}{2} + v_o\sin\theta = 0$$

$$t = 0 \quad \text{or} \quad \boxed{t = \frac{2v_o\sin\theta}{g}}$$

Now plug this nonzero time into $x(t)$ to determine how far the ball travels while it's in the air.

$$x\left(\frac{2v_o\sin\theta}{g}\right) = v_o\left(\frac{2v_o\sin\theta}{g}\right)\cos\theta$$

Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$, the ball travels eastward a distance

$$\boxed{x\left(\frac{2v_o\sin\theta}{g}\right) = \frac{v_o^2\sin 2\theta}{g}.}$$